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A superspace embedding of the Wess–Zumino model

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Abstract. We embed the Wess–Zumino (WZ) model in a wider superspace than the one described by chiral and anti-chiral superfields.

1 Introduction

There is a systematic and interesting formalism for embedding, developed by Batalin, Fradkin, Fradkina, and Tyutin (BFFT) [1], where theories with second-class constraints [2] are transformed into more general (gauge) theories where all constraints become first class. The transformation of constraints from second to first class is achieved after extending the phase space by means of auxiliary variables under the general rule that there is one pair of canonical variables for each second-class constraint. The method is iterative and can stop in the first step [3] or can go on indefinitely [4,5]. In any case, after all constraints have been transformed into first class, it is necessary to look for the Hamiltonian corresponding to this new theory. The method also permits us to obtain any involutive quantity that has zero Poisson brackets with all the constraints. The embedding Hamiltonian can be obtained in this way, starting from the initial canonical Hamiltonian and iteratively calculating the corresponding corrections.

There is another manner to obtain an embedding Hamiltonian, which consists in still using the BFFT method to obtain involutive coordinates [4]. The canonical Hamiltonian is then rewritten in terms of these new coordinates that automatically give it the involutive condition. Of course, the embedding Hamiltonians obtained from these two different ways are not necessarily the same, even though they necessarily have to describe the same physics (these Hamiltonians must differ by first-class constraints only). But the important point is that for some specific theory there may exist more than one possible embedding. It is also opportune to say, on the other hand, that there are theories which cannot be embedded [6].

One of the interesting long-standing problems that the BFFT method could be used for to address is the covariant quantization of superparticles and superstrings. However, this problem has been solved in a still different embedding procedure [7]. In fact, we realize that the meaning of embedding in field theory can be taken wider than as in the cases described by the BFFT method. The important point to be emphasized is that the embedding theory must contain all the physics of the embedded one.

We would like to address in the present paper this point of view of considering the embedding procedure in a wider sense. We concentrate on the WZ model [8] in superfield language [9,10]. Conventionally, the WZ model is always developed in terms of chiral and anti-chiral superfields, which are examples of an irreducible representation. We shall consider here a kind of embedding where we describe the WZ model by using a more general superfield representation. Contrarily to the bosonic nature of the chiral and anti-chiral superfields, the general superfield we have to use is fermionic. We shall see that there are two possible terms to figure in the Lagrangian with a relative parameter between them. We also show that for a special value of this parameter, the embedding theory exhibits a kind of gauge symmetry relating all the fields of the theory. This is a consistent result because a natural characteristic of the embedding theory is that is has more symmetries than the embedded one.

This paper is organized as follows. In Sect. 2 we discuss the general formalism. The embedding is achieved in Sect. 3. In Sect. 4 we analyze the question of gauge invariance. We devote Sect. 5 to concluding remarks and present an appendix mainly to list some identities used in this paper.

2 General formalism

In order to fix the notation and make future comparisons, let us write down the general form of the real and scalar superfield,

$$\begin{aligned}\Phi(x,\theta) = {} & A(x) + \bar\theta\psi(x) + \frac{1}{2}\,\bar\theta\theta\,B(x)\\ & + \frac{\mathrm{i}}{2}\,\bar\theta\gamma_5\theta\,C(x) + \frac{1}{2}\,\bar\theta\gamma^\mu\gamma_5\theta\,A_\mu(x)\\ & + \frac{1}{2}\bar\theta\theta\bar\theta\lambda(x) + \frac{1}{4}(\bar\theta\theta)^2 D(x). \qquad (2.1)\end{aligned}$$

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Here, all the spinors are of Majorana type and are in the Majorana representation (their components are real). We observe that it contains eight bosonic and eight fermionic degrees of freedom. We are going to work in four component notation for the spinor fields. In the appendix, we give more details about the notation and conventions we are using and list some useful identities.

The irreducible positive and negative chiral superfields that contain just four component fields are given by

$$\begin{aligned}\Phi_+(x,\theta) &= \phi(x) + \frac{\mathrm{i}}{2}\bar\theta\gamma^\mu\gamma_5\theta\partial_\mu\phi - \frac{1}{8}(\bar\theta\theta)^2\Box\phi \\ &+ \frac{1}{2}\bar\theta(1+\gamma_5)\psi(x) - \frac{\mathrm{i}}{4}\bar\theta\theta\bar\theta\gamma^\mu(1+\gamma_5)\partial_\mu\psi \\ &+ \frac{1}{4}\bar\theta(1+\gamma_5)\theta F(x), \end{aligned} \qquad (2.2)$$

$$\begin{aligned}\Phi_-(x,\theta) &= \phi^*(x) - \frac{\mathrm{i}}{2}\bar\theta\gamma^\mu\gamma_5\theta\partial_\mu\phi^* - \frac{1}{8}(\bar\theta\theta)^2\Box\phi^* \\ &+ \frac{1}{2}\bar\theta(1-\gamma_5)\psi(x) - \frac{\mathrm{i}}{4}\bar\theta\theta\bar\theta\gamma^\mu(1-\gamma_5)\partial_\mu\psi \\ &+ \frac{1}{4}\bar\theta(1-\gamma_5)\theta F^*(x). \end{aligned} \qquad (2.3)$$

The WZ model [8,10] is directly obtained (up to some general constant factor) from an action given by the product of positive and negative chiral superfields, $S = \int \mathrm{d}^4x\mathrm{d}^4\theta\Phi_+\Phi_-$.

We first observe that the formulation of a supersymmetric theory, using general superfields and that containing the WZ model as a particular case, cannot be done in terms of covariant derivatives over the scalar superfield. This is so because it would violate the correct mass dimension of the superfield Lagrangian, that should be two. The correct way is to start from a fermionic superfield, whose general form reads

$$\begin{aligned}\Psi_\alpha(x,\theta) &= \chi_\alpha(x) + \theta_\alpha\phi(x) + \frac{1}{2}\bar\theta\theta\psi_\alpha(x) \\ &+ \frac{\mathrm{i}}{2}\bar\theta\gamma_5\theta\lambda_\alpha(x) + \frac{1}{2}\bar\theta\gamma^\mu\gamma_5\theta\psi_{\mu\alpha}(x) \\ &+ \frac{1}{2}\bar\theta\theta\theta_\alpha F(x) + \frac{1}{4}(\bar\theta\theta)^2\eta_\alpha(x). \end{aligned} \qquad (2.4)$$

Consequently, the form of $\bar\Psi_\alpha(x,\theta)$ is

$$\begin{aligned}\bar\Psi_\alpha(x,\theta) &= \bar\chi_\alpha(x) + \bar\theta_\alpha\phi^*(x) + \frac{1}{2}\bar\theta\theta\bar\psi_\alpha(x) \\ &+ \frac{\mathrm{i}}{2}\bar\theta\gamma_5\theta\bar\lambda_\alpha(x) + \frac{1}{2}\bar\theta\gamma^\mu\gamma_5\theta\bar\psi_{\mu\alpha}(x) \\ &+ \frac{1}{2}\bar\theta\theta\bar\theta_\alpha F^*(x) + \frac{1}{4}(\bar\theta\theta)^2\bar\eta_\alpha(x). \end{aligned} \qquad (2.5)$$

If we consider the fermionic superfield with mass dimension 1/2, the mass dimensions of the component fields are

$$\begin{aligned}&[\chi] = \frac{1}{2}, \quad [\phi] = 1, \quad [\psi] = [\lambda] = [\psi_\mu] = \frac{3}{2}, \\ &[F] = 2, \quad [\eta] = \frac{5}{2}. \end{aligned} \qquad (2.6)$$

Notice that ϕ, ψ, and F actually have the same mass dimensions as the corresponding fields of the WZ model.

The supersymmetry transformations of the component fields can be directly obtained by the general supersymmetry transformation relation

$$\delta\Psi_\alpha = (\bar\xi Q)\Psi_\alpha, \qquad (2.7)$$

which leads to

$$\begin{aligned}\delta\chi_\alpha &= \xi_\alpha\phi, \\ \delta\phi &= -\frac{\mathrm{i}}{4}\bar\xi\partial\!\!\!/\chi - \frac{1}{4}\bar\xi\psi - \frac{\mathrm{i}}{4}\bar\xi\gamma_5\lambda + \frac{1}{4}\bar\xi\gamma_5\gamma^\mu\psi_\mu, \\ \delta\psi_\alpha &= \frac{\mathrm{i}}{2}(\partial\!\!\!/\phi\xi)_\alpha + \frac{1}{2}\xi_\alpha F, \\ \delta\lambda_\alpha &= \frac{1}{2}(\gamma_5\partial\!\!\!/\phi\xi)_\alpha + \frac{\mathrm{i}}{2}(\gamma_5\xi)_\alpha F, \\ \delta\psi_{\mu\alpha} &= \frac{\mathrm{i}}{2}(\gamma_5\gamma_\mu\partial\!\!\!/\phi\xi)_\alpha - \frac{1}{2}(\gamma_5\gamma_\mu\xi)_\alpha F, \\ \delta F &= \frac{\mathrm{i}}{4}\bar\xi\partial\!\!\!/\psi + \frac{1}{4}\bar\xi\gamma_5\partial\!\!\!/\lambda + \frac{\mathrm{i}}{4}\bar\xi\gamma^\mu\gamma^\nu\gamma_5\partial_\mu\psi_\nu - \frac{1}{4}\bar\xi\eta, \\ \delta\eta_\alpha &= \frac{\mathrm{i}}{2}(\partial\!\!\!/ F\xi)_\alpha. \end{aligned} \qquad (2.8)$$

The form of the charge operators as well as the derivative operators are given in the appendix. We observe that the usual transformations of the component fields of the WZ model are embodied in (2.8).

Before going on, it is opportune to make a comment about the number of bosonic and fermionic degrees of freedom that appear in (2.4) and (2.5). At first sight, they are not the same. There are thirty-two fermionic degrees of freedom and apparently much less bosonic ones. What happens is that the bosonic quantities ϕ and F are not representing just single fields. In stead of the quantity $\theta_\alpha\phi$, we must more generically read [11]

$$\begin{aligned}\theta_\alpha\phi \longrightarrow\ & \theta_\alpha\phi + (\gamma_5\theta)_\alpha\tilde\phi + (\gamma^\mu\theta)_\alpha A_\mu \\ &+ (\gamma_5\gamma^\mu\theta)_\alpha\tilde A_\mu + \frac{1}{2}(\sigma^{\mu\nu}\theta)_\alpha B_{\mu\nu}, \end{aligned} \qquad (2.9)$$

which corresponds to sixteen degrees of freedom, and the same occurs for the term with F.

3 The embedding theory

The most general supersymmetric action whose Lagrangian density is expressed in terms of the spinor superfields (without involving high derivatives and nonlocal terms) has the form

$$S = \int \mathrm{d}^4x\mathrm{d}^4\theta(\bar D_\alpha\Psi_\alpha D_\beta\bar\Psi_\beta + a\bar D_\alpha\Psi_\beta D_\alpha\bar\Psi_\beta), \qquad (3.1)$$

where a is a relative normalization parameter that shall be conveniently fixed. The effective Lagrangian density is the $(\bar\theta\theta)^2$ component of the general Lagrangian density that appears in (3.1). Denoting the effective Lagrangian density by $\mathcal{L}$ we obtain, after a long algebraic calculation,

$$\mathcal{L} = \left(2a + \frac{1}{2}\right)\bar\psi\eta - \left(2a + \frac{1}{2}\right)\bar\lambda\partial^\mu\psi_\mu$$

$$
\begin{aligned}
&-\left(a+\frac{1}{4}\right)\bar{\psi}\Box\chi+(a-1)FF^*-a\phi\Box\phi^*\\
&+\frac{\mathrm{i}}{4}\bar{\psi}\partial\!\!\!/\psi+\frac{1}{2}\bar{\psi}\gamma_5\partial\!\!\!/\lambda+\frac{\mathrm{i}}{2}\bar{\psi}\gamma_5\partial^\mu\psi_\mu\\
&+\frac{\mathrm{i}}{2}\bar{\lambda}\gamma_5\eta+\frac{\mathrm{i}}{4}\bar{\lambda}\partial\!\!\!/\lambda+\frac{1}{2}\bar{\psi}_\mu\gamma^\mu\gamma_5\eta\\
&+\frac{\mathrm{i}}{4}\bar{\psi}_\mu\gamma^\mu\gamma^\nu\partial\!\!\!/\psi_\nu+\frac{\mathrm{i}}{2}\bar{\eta}\partial\!\!\!/\chi-\frac{\mathrm{i}}{4}\bar{\chi}\gamma_5\Box\lambda\\
&+\frac{1}{4}\bar{\chi}\gamma_5\gamma^\mu\gamma^\nu\partial\!\!\!/\partial_\mu\psi_\nu. \qquad (3.2)
\end{aligned}
$$

We notice that the bosonic quantities ϕ and F do not mix with any fermionic fields and their equations of motion are the usual ones that appear in the WZ model (up to the generic scaling parameter a). Further, we also observe that the relative parameter a cannot be one because it would rule out the term in FF^*. Concerning the equations of motion for the fermionic fields, we have

$$
\begin{aligned}
&2(4a+1)\eta-(4a+1)\Box\chi+2\mathrm{i}\partial\!\!\!/\psi+2\gamma_5\partial\!\!\!/\lambda\\
&\quad+2\mathrm{i}\gamma_5\partial^\mu\psi_\mu=0, && (3.3)\\
&(4a+1)\psi+\mathrm{i}\gamma_5\lambda-\gamma_5\gamma^\mu\psi_\mu+\mathrm{i}\partial\!\!\!/\chi=0, && (3.4)\\
&2(4a+1)\partial^\mu\psi_\mu+2\gamma_5\partial\!\!\!/\psi-2\mathrm{i}\gamma_5\eta\\
&\quad-2\mathrm{i}\partial\!\!\!/\lambda+\mathrm{i}\gamma_5\Box\chi=0, && (3.5)\\
&2(4a+1)\partial^\mu\lambda-2\mathrm{i}\gamma_5\partial^\mu\psi+2\gamma^\mu\gamma_5\eta\\
&\quad+\mathrm{i}(\gamma^\mu\gamma^\rho\gamma^\nu+\gamma^\nu\gamma^\mu\gamma^\rho)\partial_\nu\psi_\rho+\gamma_5\gamma^\nu\gamma^\mu\partial\!\!\!/\partial_\nu\chi=0, && (3.6)\\
&(4a+1)\Box\psi-2\mathrm{i}\partial\!\!\!/\eta+\mathrm{i}\gamma_5\Box\lambda-\gamma_5\gamma^\mu\gamma^\nu\partial\!\!\!/\partial_\mu\psi_\nu=0. && (3.7)
\end{aligned}
$$

These respectively correspond to the variations with respect to $\bar{\psi}$, $\bar{\eta}$, $\bar{\lambda}$, $\bar{\psi}_\mu$, and $\bar{\chi}$, and they were used in the identities (A.10)–(A.16).

As has been emphasized, the procedure of embedding must not affect the physics we already know for the initial theory. We can verify that this is actually the case by combining these equations to obtain equations of motion for each component field. For example, by using (3.4) and (3.5) we eliminate λ and η from the remaining equations. The result is

$$
\begin{aligned}
&4a\partial\!\!\!/\psi-2(2a+1)\gamma_5\partial^\mu\psi_\mu+\gamma_5\partial\!\!\!/\gamma^\mu\psi_\mu+\mathrm{i}\Box\chi=0, && (3.8)\\
&2(2a-1)\partial\!\!\!/\psi+4\gamma_5\partial^\mu\psi_\mu+\gamma_5\partial\!\!\!/\gamma^\mu\psi_\mu+\mathrm{i}\Box\chi=0, && (3.9)\\
&\partial\!\!\!/\psi+\gamma_5\partial^\mu\psi_\mu=0. && (3.10)
\end{aligned}
$$

The analysis of these equations shows us that for $a=-2$ they are not independent, and, consequently, the solution for the equations of motion is not unique. On the other hand, for $a\neq-2$ we unambiguously have

$$\partial\!\!\!/\psi=0, \qquad (3.11)$$

$$\partial^\mu\psi_\mu=0, \qquad (3.12)$$

$$\Box\chi-\mathrm{i}\gamma_5\partial\!\!\!/\gamma^\mu\psi_\mu=0. \qquad (3.13)$$

Introducing these results into (3.4) and (3.5), one obtains the following equations involving λ and η:

$$\partial\!\!\!/\lambda=0, \qquad (3.14)$$

$$\Box\chi-2\eta=0. \qquad (3.15)$$

Remember that the equations of motion for ϕ and F can be directly obtained from the Lagrangian (3.2) (since $a\neq1$), and we have seen above that the equation for ψ can be unambiguously obtained by the combination of the set given by (3.3)–(3.7) (since $a\neq-2$). The equations are the same as in the WZ model. Consequently the supersymmetric embedding of the model has succeeded by means of the Lagrangian (3.2) if we respect the restriction that the parameter a has to be different from 1 and -2.

4 Gauge symmetry

In the previous analysis we have seen that the supersymmetric embedding of the WZ model starting from the Lagrangians (3.1) and (3.2) does not completely fix the value of the relative parameter a. In this section we are going to see that the embedding theory exhibits a kind of gauge symmetry that only occurs for a specific value of the parameter a compatible with the previous restrictions. To see this we take a generic variation of the Lagrangian (3.2),

$$
\begin{aligned}
\delta\mathcal{L}=\;&\delta\bar{\psi}\Big[\left(2a+\frac{1}{2}\right)\eta-\left(a+\frac{1}{4}\right)\Box\chi+\frac{\mathrm{i}}{2}\partial\!\!\!/\psi\\
&\quad+\frac{1}{2}\gamma_5\partial\!\!\!/\lambda+\frac{\mathrm{i}}{2}\gamma_5\partial^\mu\psi_\mu\Big]\\
&+\delta\bar{\eta}\Big[\left(2a+\frac{1}{2}\right)\psi+\frac{\mathrm{i}}{2}\gamma_5\lambda-\frac{1}{2}\gamma_5\gamma^\mu\psi_\mu+\frac{\mathrm{i}}{2}\partial\!\!\!/\chi\Big]\\
&-\delta\bar{\lambda}\Big[\left(2a+\frac{1}{2}\right)\partial^\mu\psi_\mu+\frac{1}{2}\gamma_5\partial\!\!\!/\psi\\
&\quad-\frac{\mathrm{i}}{2}\gamma_5\eta-\frac{\mathrm{i}}{2}\partial\!\!\!/\lambda+\frac{\mathrm{i}}{4}\gamma_5\Box\chi\Big]\\
&+\delta\bar{\psi}_\mu\Big[\left(2a+\frac{1}{2}\right)\partial^\mu\lambda-\frac{\mathrm{i}}{2}\gamma_5\partial^\mu\psi+\frac{1}{2}\gamma^\mu\gamma_5\eta\\
&\quad+\frac{\mathrm{i}}{4}(\gamma^\mu\gamma^\rho\gamma^\nu+\gamma^\nu\gamma^\mu\gamma^\rho)\partial_\nu\psi_\rho+\frac{1}{4}\gamma_5\gamma^\nu\gamma^\mu\partial\!\!\!/\partial_\nu\chi\Big]\\
&-\delta\bar{\chi}\Big[\left(a+\frac{1}{4}\right)\Box\psi-\frac{\mathrm{i}}{2}\partial\!\!\!/\eta+\frac{\mathrm{i}}{4}\gamma_5\Box\lambda\\
&\quad-\frac{1}{4}\gamma_5\gamma^\mu\gamma^\nu\partial\!\!\!/\partial_\mu\psi_\nu\Big]+\delta A^*\Box A+\delta F^*F. \qquad (4.1)
\end{aligned}
$$

Looking at the equation of motion (3.13), it suggests that a possible gauge transformation for $\bar{\psi}_\mu$ should have the form

$$\delta\bar{\psi}_\mu(x)=\bar{\alpha}(x)\gamma_\mu\gamma_5, \qquad (4.2)$$

where $\alpha(x)$ is a Majorana spinor that plays the role of a gauge parameter. Even though we have considered the form of the equation of motion in order to have insight in the kind of gauge transformation, we emphasize that we are going to work off-shell.

Keeping in mind the mass dimensions of the fields that appear in (4.1), we infer that the gauge transformations for the remaining fields should be

$$\delta\bar{\psi}(x)=b\bar{\alpha}(x),$$

$$
\begin{aligned}
\delta\bar\eta(x) &= c\partial_\mu\bar\alpha(x)\gamma^\mu,\\
\delta\bar\lambda(x) &= d\bar\alpha(x)\gamma_5,\\
\delta\bar\chi(x) &= e\frac{1}{\Box}\partial_\mu\bar\alpha(x)\gamma^\mu,\\
\delta A^*(x) &= 0,\\
\delta F^*(x) &= 0,
\end{aligned}
\tag{4.3}
$$

where b, c, d, and e are parameters to be conveniently fixed. Replacing (4.2) and (4.3) into (4.1) we see that the necessary condition to get the symmetry is

$$
\begin{aligned}
2b + 2\mathrm{i}c + 2\mathrm{i}d - \mathrm{i}e &= 8a + 2,\\
2b + (4a+1)2\mathrm{i}c + 2\mathrm{i}d - (4a+1)\mathrm{i}e &= 2,\\
(4a+1)b + \mathrm{i}d - \mathrm{i}e &= 4,\\
(4a+1)b + 2\mathrm{i}c + \mathrm{i}d &= -2.
\end{aligned}
\tag{4.4}
$$

These correspond to the coefficients of $\bar\alpha\gamma_5\partial\!\!\!/\lambda$, $\bar\alpha\partial\!\!\!/\psi$, $\bar\alpha\eta$, and $\bar\alpha\Box\chi$, respectively. There is still another equation to be verified which is related to the field ψ_μ, namely

$$
\begin{aligned}
&(-6\mathrm{i} + 2c + e)\bar\alpha\gamma_5\gamma^\mu\partial\!\!\!/\psi_\mu\\
&\quad+[4\mathrm{i} + 2b\mathrm{i} - 4c - 2d(4a+1)]\bar\alpha\gamma_5\partial^\mu\psi_\mu = 0,
\end{aligned}
\tag{4.5}
$$

where one cannot infer any conclusion for the coefficients of $\bar\alpha\gamma_5\gamma^\mu\partial\!\!\!/\psi_\mu$ and $\bar\alpha\gamma_5\partial^\mu\psi_\mu$ because these terms are not independent.

Considering the set given by (4.4), one can solve it to express b, c, d, and e in terms of a. The result is $b = -1$, $c = 2\mathrm{i}$, $d = -\mathrm{i}(4a+3)$, and $e = 2\mathrm{i}$ (it is important to mention that this solution exists only if $a \neq 0$). Introducing now this result into (4.5), we get a providential cancellation of the first term. The second one becomes

$$
a(a+1)\bar\alpha\gamma_5\partial^\mu\psi_\mu = 0. \tag{4.6}
$$

Since a cannot be zero, we see that the symmetry given by (4.2) and (4.3) fixes the parameter a to -1 [this value is compatible with all the previous boundary conditions and we also notice that it does not rule out any term of the initial Lagrangian (3.2)].

5 Conclusion

In this work we have embedded the WZ model in a wider superspace than the one described by chiral and anti-chiral superfields. We have shown that it is appropriate just to use the fermionic general superfield, and the consistency condition of the embedding is verified by showing that the same equations of motion as in the WZ model are among the equations of motion of the general model since the relative parameter that appears in the two terms of the Lagrangian is different from -2 and 1. Finally, we have also shown that the embedding theory has a kind of gauge symmetry. This symmetry permits us to fix the relative parameter, and its value is compatible with the restriction above.

A remaining question concerns other gauge symmetries embodied in the Lagrangian density (3.2). This can be dealt with in the Hamiltonian formalism by analyzing the first-class constraints of the theory [12]. However, since fermionic fields appear in a completely coupled way in the Lagrangian, it is not a simple task to envisage what are all the independent first-class constraints we can build up. This subject is presently under study and possible results shall be reported elsewhere.

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A Appendix

In this appendix, we present the notation, conventions and the main identities used throughout this paper. The gamma matrices satisfy the usual relations $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$ and $\gamma^\mu = \gamma^0\gamma^{\mu\dagger}\gamma^0$. We adopt the metric convention $\eta^{\mu\nu} = \mathrm{diag.}(1,-1,-1,-1)$. We take the completely antisymmetric tensor $\epsilon^{\mu\nu\rho\lambda}$ given by $\epsilon^{0123} = 1$. The matrices γ_5 and $\sigma^{\mu\nu}$ are defined by

$$
\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3, \tag{A.1}
$$

$$
\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu,\gamma^\nu]. \tag{A.2}
$$

Let us list below some useful identities involving gamma matrices:

$$
\gamma^\mu\gamma^\nu\gamma^\rho = \eta^{\mu\nu}\gamma^\rho - \eta^{\mu\rho}\gamma^\mu + \eta^{\nu\rho}\gamma^\mu - \mathrm{i}\epsilon^{\mu\nu\rho\lambda}\gamma_5\gamma_\lambda, \tag{A.3}
$$

$$
\gamma_5\gamma^\mu\gamma^\nu = \eta^{\mu\nu}\gamma_5 + \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\sigma_{\rho\lambda}, \tag{A.4}
$$

$$
\gamma_5\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\lambda}\sigma_{\rho\lambda}, \tag{A.5}
$$

$$
\gamma^\mu\sigma^{\rho\lambda} = \frac{\mathrm{i}}{2}(\eta^{\mu\rho}\gamma^\lambda - \eta^{\mu\lambda}\gamma^\rho) + \frac{1}{2}\epsilon^{\mu\rho\lambda\nu}\gamma_5\gamma_\nu, \tag{A.6}
$$

$$
\sigma^{\mu\nu}\gamma^\rho = \frac{\mathrm{i}}{2}(\eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu) + \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\gamma_5\gamma_\lambda, \tag{A.7}
$$

$$
\begin{aligned}
\sigma^{\mu\nu}\sigma^{\rho\lambda} &= \frac{\mathrm{i}}{4}\epsilon^{\mu\nu\rho\lambda}\gamma_5 + \frac{1}{4}(\eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\mu\lambda}\eta^{\nu\rho})\\
&\quad- \frac{\mathrm{i}}{2}(\eta^{\mu\rho}\eta^{\nu\alpha}\eta^{\lambda\beta} + \eta^{\nu\lambda}\eta^{\mu\alpha}\eta^{\rho\beta} - \rho \leftrightarrow \lambda).
\end{aligned}
\tag{A.8}
$$

Further,

$$
\begin{aligned}
\mathrm{tr}\gamma^\mu\gamma^\nu &= 4\eta^{\mu\nu},\\
\mathrm{tr}\gamma_5 &= 0,\\
\mathrm{tr}\gamma_5\gamma^\mu\gamma^\nu &= 0,\\
\mathrm{tr}\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda &= 4\mathrm{i}\epsilon^{\mu\nu\rho\lambda},\\
\mathrm{tr}\sigma^{\mu\nu}\sigma^{\rho\lambda} &= 4(\eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\mu\lambda}\eta^{\nu\rho}).
\end{aligned}
\tag{A.9}
$$

Considering ψ and χ as Majorana spinors, we also have

$$
\bar\psi\chi = \bar\chi\psi, \tag{A.10}
$$

$$
\bar\psi\gamma_5\chi = \bar\chi\gamma_5\psi, \tag{A.11}
$$

$$
\bar\psi\gamma^\mu\gamma_5\chi = \bar\chi\gamma^\mu\gamma_5\psi, \tag{A.12}
$$

$$
\bar\psi\gamma^\mu\chi = -\bar\chi\gamma^\mu\psi, \tag{A.13}
$$

$$
\bar\psi\sigma^{\mu\nu}\chi = -\bar\chi\sigma^{\mu\nu}\psi, \tag{A.14}
$$

$$
\bar\psi\gamma^\mu\gamma^\nu\chi = \bar\chi\gamma^\nu\gamma^\mu\psi. \tag{A.15}
$$

Using the relations (A.3)–(A.15), we obtain the additional relations

$$\begin{aligned}
\bar\psi\gamma_5\sigma^{\mu\nu}\chi &= -\bar\chi\gamma_5\sigma^{\mu\nu}\psi,\\
\bar\psi\gamma_5\gamma^\mu\gamma^\nu\chi &= \bar\chi\gamma_5\gamma^\nu\gamma^\mu\psi,\\
\bar\psi\gamma^\mu\gamma^\nu\gamma^\rho\chi &= -\bar\chi\gamma^\rho\gamma^\nu\gamma^\mu\psi,\\
\bar\psi\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\chi &= \bar\chi\gamma_5\gamma^\rho\gamma^\nu\gamma^\mu\psi,\\
\bar\psi\gamma_5\gamma^\rho\sigma^{\mu\nu}\chi &= -\bar\chi\gamma_5\sigma^{\mu\nu}\gamma^\rho\psi,\\
\bar\psi\sigma^{\mu\nu}\gamma^\rho\gamma^\lambda\chi &= \bar\chi\gamma^\lambda\gamma^\rho\sigma^{\mu\nu}\psi,\\
\bar\psi\gamma_5\sigma^{\mu\nu}\gamma^\alpha\sigma^{\rho\lambda}\chi &= \bar\chi\gamma_5\sigma^{\rho\lambda}\gamma^\alpha\sigma^{\mu\nu}\psi.
\end{aligned}\tag{A.16}$$

The Fierz identity reads

$$\frac{1}{4}(\Gamma^A)_{\alpha\beta}(\Gamma_A)_{\sigma\rho} = \delta_{\alpha\rho}\delta_{\beta\sigma},\tag{A.17}$$

where Γ^A is generically representing the independent matrices: $\Gamma^1 = 1$, Γ^2 to $\Gamma^5 = \gamma^\mu$, $\Gamma^6 = \gamma_5$, Γ^7 to $\Gamma^{10} = \gamma^\mu\gamma_5$, Γ^{11} to $\Gamma^{16} = \sigma^{\mu\nu}$. Concerning Γ_A, the corresponding relations are almost trivial; we just have to notice the inverse order between γ_5 and γ_μ from Γ_7 to $\Gamma_{10} = \gamma_5\gamma_\mu$. Using the Fierz identity, we obtain

$$\begin{aligned}
\theta_\alpha\bar\theta_\beta &= -\frac{1}{4}\delta_{\alpha\beta}\bar\theta\theta - \frac{1}{4}\gamma_{5\alpha\beta}\bar\theta\gamma_5\theta\\
&\quad - \frac{1}{4}(\gamma^\mu\gamma_5)_{\alpha\beta}\bar\theta\gamma_5\gamma_\mu\theta,\\
\bar\theta\gamma_5\theta\bar\theta_\alpha &= -\bar\theta\theta(\bar\theta\gamma_5)_\alpha,\\
\theta_\alpha\bar\theta\gamma_5\theta &= -(\gamma_5\theta)_\alpha\bar\theta\theta,\\
\bar\theta\gamma_5\gamma_\mu\theta\bar\theta_\alpha &= -\bar\theta\theta(\bar\theta\gamma_5\gamma_\mu)_\alpha,\\
\theta_\alpha\bar\theta\gamma_5\gamma_\mu\theta &= -(\gamma_5\gamma_\mu\theta)_\alpha\bar\theta\theta,\\
\bar\theta\gamma_5\theta\bar\theta\gamma_5\theta &= -(\bar\theta\theta)^2,\\
\bar\theta\gamma^\mu\gamma_5\theta\bar\theta\gamma^\nu\gamma_5\theta &= \eta^{\mu\nu}(\bar\theta\theta)^2,\\
\bar\theta\theta\bar\theta\gamma_5\theta &= 0,\\
\bar\theta\theta\bar\theta\gamma^\mu\gamma_5\theta &= 0,\\
\bar\theta\gamma_5\theta\bar\theta\gamma^\mu\gamma_5\theta &= 0.
\end{aligned}\tag{A.18}$$

The supersymmetry charge and derivative operators are defined by

$$\begin{aligned}
Q_\alpha &= \frac{\partial}{\partial\bar\theta_\alpha} + \mathrm{i}(\gamma^\mu\theta)_\alpha\partial_\mu,\\
\bar Q_\alpha &= -\frac{\partial}{\partial\theta_\alpha} - \mathrm{i}(\bar\theta\gamma^\mu)_\alpha\partial_\mu,\\
D_\alpha &= \frac{\partial}{\partial\bar\theta_\alpha} - \mathrm{i}(\gamma^\mu\theta)_\alpha\partial_\mu,\\
\bar D_\alpha &= -\frac{\partial}{\partial\theta_\alpha} + \mathrm{i}(\bar\theta\gamma^\mu)_\alpha\partial_\mu.
\end{aligned}\tag{A.19}$$

Positive and negative chiralities are defined as

$$\theta_\pm = \frac{1}{2}(1\pm\gamma_5)\theta;\tag{A.20}$$

consequently,

$$\begin{aligned}
\frac{\partial}{\partial\theta_\pm}\theta_\pm &= \frac{1}{2}(1\pm\gamma_5),\\
\frac{\partial}{\partial\bar\theta_\pm}\bar\theta_\pm &= \frac{1}{2}(1\mp\gamma_5),\\
\frac{\partial}{\partial\theta_\pm}\bar\theta_\pm &= \frac{1}{2}(1\pm\gamma_5)\gamma^0,\\
\frac{\partial}{\partial\bar\theta_\pm}\theta_\pm &= \frac{1}{2}(1\mp\gamma_5)\gamma^0,\\
\frac{\partial}{\partial\theta_\pm}\theta_\mp &= 0,\\
\frac{\partial}{\partial\bar\theta_\pm}\bar\theta_\mp &= 0,\\
\frac{\partial}{\partial\theta_\pm}\bar\theta_\mp &= 0,\\
\frac{\partial}{\partial\bar\theta_\pm}\theta_\mp &= 0.
\end{aligned}\tag{A.21}$$

The positive and negative chiral superfields satisfy

$$D_\mp\Phi_\pm = 0.\tag{A.22}$$

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